Hyperbolic Functions

This topic is NOT part of the AP curriculum, however, it is part of a typical College Calculus course.

Hyperbolic Functions

\[
\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{or} \quad \frac{1}{2} e^x + \frac{1}{2} e^{-x}
\]

\[
\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{or} \quad \frac{1}{2} e^x - \frac{1}{2} e^{-x}
\]

Properties

- \( \cosh 0 = 1 \)
- \( \sinh 0 = 0 \)
- \( \cosh^2 x - \sinh^2 x = 1 \)
- \( \cosh(a + b) = \cosh a \cosh b + \sinh a \sinh b \)
- \( \sinh(a + b) = \sinh a \cosh b + \cosh a \sinh b \)

Derivatives

\[
\frac{d}{dx} (\sinh x) = \cosh x \quad \frac{d}{dx} (\cosh x) = \sinh x
\]

\[
\frac{d}{dx} (\tanh x) = \sec^2 x \quad \frac{d}{dx} (\coth x) = -\csc^2 x
\]

\[
\text{OR} = \frac{1}{\cosh^2 x} \quad \text{OR} = -\frac{1}{\sinh^2 x}
\]

\[
\frac{d}{dx} (\sec x) = -\sec x \tanh x \quad \frac{d}{dx} (\csc x) = -\csc x \cot x
\]

\[
\frac{d}{dx} (\sinh^{-1} x) = \frac{1}{\sqrt{x^2 + 1}} \quad \frac{d}{dx} (\cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}} \quad \frac{d}{dx} (\tanh^{-1} x) = \frac{1}{1 - x^2}
\]
1. Use the exponential formula for hyperbolic functions to show that the properties are true.
   a) \( \sinh 0 = 0 \)  
   b) \( \cosh (-x) = \cosh x \)

2. Based on your answer from 1b), is \( \cosh x \) an odd or even function?

3. Prove that \( \sinh x \) is an odd function.

4. Use the exponential formula for hyperbolic functions to simplify the expressions.
   a) \( \sinh (\ln t) \)  
   b) \( \tanh (\ln t) \)
5. Use the exponential formula for hyperbolic functions to show that the properties to evaluate each limit.

a) \( \lim_{x \to \infty} \frac{\cosh(2x)}{\sinh(3x)} \)  

b) \( \lim_{x \to \infty} \frac{e^{2x}}{\sinh(2x)} \)  

c) \( \lim_{x \to \infty} \frac{\sinh(x^2)}{\cosh(x^2)} \)

6. Find the derivatives.

a) \( y = \sinh(2x) \)  

b) \( f(t) = \cosh(\sinh t) \)

c) \( g(\theta) = \ln(\cosh(1 + \theta)) \)  

d) \( y = \cosh^2 t + \cosh(e^t) \)
7. Show that \( \frac{d}{dx} (\tanh x) = \text{sech}^2 x \)

8. Show that if \( \sinh^{-1} x = \ln \left( x + \sqrt{x^2 + 1} \right) \),

\[ \text{then } \frac{d}{dx} (\sinh^{-1} x) = \frac{1}{\sqrt{x^2 + 1}}. \]

9. The Saint Louis arch can be approximated by using a function of the form \( y = b - a \cosh(x/a) \). Putting the origin on the ground in the center of the arch and the \( y \)-axis upward, find an approximate equation for the arch given the dimensions shown in Figure 3.37. (In other words, find \( a \) and \( b \).)
10. Consider the function $y = \sinh(2 - \sqrt{x})$ for $x > 0$.
   
   a) Find the equation of the tangent line to the graph of $y$ at $x = 4$.

   b) Use the equation of the tangent line at $x = 4$ to estimate $y(4.1)$.

   c) Determine if the estimate $y(4.1)$ is an overestimate or underestimate. Justify your answer.