Semester 2 Unit 3 – 3- 1 WKST Name:______________________________

Separation of Variables
No Graphing Calculator is allowed for these problems.

Use Separation of Variables to solve the Differential Equations.
Note: not every function is defined for all values of x and y.

1. $\frac{dy}{dx} = \frac{6}{y^3}$

2. $\frac{dy}{dx} = \frac{x^2}{y + y^2}$

3. $\frac{dy}{dx} = y \cos x$

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4. [2001 AB – NO Calculator] The function f is differentiable for all real numbers. The point (3 , $\frac{1}{4}$) is on the graph of $y = f(x)$, and the slope at each point (x , y) on the graph is given by $\frac{dy}{dx} = y^2(6 - 2x)$.

(a) Find $\frac{d^2y}{dx^2}$ and evaluate it at the point (3 , $\frac{1}{4}$).

(b) Find $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = y^2(6 - 2x)$ with the initial condition $f(3) = \frac{1}{4}$.

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5. [2000 AB – NO Calculator] Consider the differential equation $\frac{dy}{dx} = \frac{3x^2}{e^{2y}}$.

(a) Find a solution $y = f(x)$ to the differentiable equation satisfying $f(0) = \frac{1}{2}$.

(b) Find the domain and range of the function f found in part (a).

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6. [1998 AB – NO Calculator] Let f be a function with $f(1) = 4$ such that for all points (x , y) on the graph of f the slope is given by $\frac{3x^2 + 1}{2y}$.

(a) Find the slope of the graph of f at the point where $x = 1$.

(b) Write an equation for the line tangent to the graph of f at $x = 1$.

[Continue with #6] Let f be a function with $f(1)=4$ such that for all points (x , y) on the graph of f the slope is given by $\frac{3x^2+1}{2y}$.

(c) Find $f(x)$ by solving the separable differential equation $\frac{dy}{dx}=\frac{3x^2+1}{2y}$ with the initial condition $f(1)=4$.

(d) Use your solution from part (c) to find $f(1.2)$.

More Diffy-Q fun!!! Use Separation of Variables to solve the Differential Equations.

7. $y'=\sqrt{x}\cos^2 y$

8. $xy+y'=5x$

9. $x^2 y' - 2y = 4$

10. $(1 + x^2)y' + y = 2y$

11. $x^2 y' - x = y'$

ANSWERS:

1) $\frac{1}{4}y^4 = 6x + C$

2) $\frac{1}{2}y^2 + \frac{1}{3}y^3 = \frac{1}{3}x^3 + C$

3) $\ln|y| = \sin x + C$

4a) $2y^3(6-2x)^2 - 2y^2$; $-\frac{1}{8}$

b) $y = (x^2 - 6x + 13)^{-1}$

5a) $y = \frac{1}{2}\ln(2x^3 + e)$

b) $D: x > \sqrt[3]{-\frac{e}{2}}$; R: all reals

6a) $\frac{1}{2}$

b) $y - 4 = \frac{1}{2}(x - 1)$

c) $y = \sqrt{x^3 + x + 14}$

d) $\sqrt{16.928}$

7) $\tan y = \frac{2}{3}x^{\frac{3}{2}} + C$

8) $-\ln|5 - y| = \frac{1}{2}x^2 + C$

9) $\frac{1}{2}\ln|4 + 2y| = -x^{-1} + C$

10) $\ln|y| = \arctan x + C$

11) $y = \frac{1}{2}\ln|x^2 - 1| + C$

Sem 2 Unit 3 – 3- 1 SCORING GUIDE

4. [2001 AB – NO Calculator] The function f is differentiable for all real numbers. The point $(3, \frac{1}{4})$ is on the graph of $y = f(x)$, and the slope at each point (x, y) on the graph is given by $\frac{dy}{dx} = y^2(6-2x)$.

(a) Find $\frac{d^2y}{dx^2}$ and evaluate it at the point $(3, \frac{1}{4})$.

(b) Find $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = y^2(6-2x)$ with the initial condition $f(3) = \frac{1}{4}$.

(a)
$$\frac{d^2y}{dx^2} = 2y\frac{dy}{dx}(6-2x) - 2y^2$$
$$= 2y^3(6-2x)^2 - 2y^2$$

$$\left.\frac{d^2y}{dx^2}\right|_{\left(3,\frac{1}{4}\right)} = 0 - 2\left(\frac{1}{4}\right)^2 = -\frac{1}{8}$$

3 :
- 2 : $\frac{d^2y}{dx^2}$
  - < −2 > product rule or chain rule error
- 1 : value at $\left(3, \frac{1}{4}\right)$

(b)
$$\frac{1}{y^2}dy = (6-2x)\,dx$$

$$-\frac{1}{y} = 6x - x^2 + C$$

$$-4 = 18 - 9 + C = 9 + C$$

$$C = -13$$

$$y = \frac{1}{x^2 - 6x + 13}$$

6 :
- 1 : separates variables
- 1 : antiderivative of dy term
- 1 : antiderivative of dx term
- 1 : constant of integration
- 1 : uses initial condition $f(3) = \frac{1}{4}$
- 1 : solves for y

Note: max 3/6 [1-1-1-0-0-0] if no constant of integration

Note: 0/6 if no separation of variables

5. [2000 AB – NO Calculator] Consider the differential equation $\frac{dy}{dx} = \frac{3x^2}{e^{2y}}$.

(a) Find a solution $y = f(x)$ to the differentiable equation satisfying $f(0) = \frac{1}{2}$.

(b) Find the domain and range of the function f found in part (a).

(a) $e^{2y}\,dy = 3x^2\,dx$

$\frac{1}{2}e^{2y} = x^3 + C_1$

$e^{2y} = 2x^3 + C$

$y = \frac{1}{2}\ln\left(2x^3 + C\right)$

$\frac{1}{2} = \frac{1}{2}\ln(0 + C); \quad C = e$

$y = \frac{1}{2}\ln(2x^3 + e)$

6 {
- 1 : separates variables
- 1 : antiderivative of dy term
- 1 : antiderivative of dx term
- 1 : constant of integration
- 1 : uses initial condition $f(0) = \frac{1}{2}$
- 1 : solves for y

Note: 0/1 if y is not a logarithmic function of x

Note: max 3/6 [1-1-1-0-0-0] if no constant of integration

Note: 0/6 if no separation of variables

(b) Domain: $2x^3 + e > 0$

$x^3 > -\frac{1}{2}e$

$x > \left(-\frac{1}{2}e\right)^{1/3} = -\left(\frac{1}{2}e\right)^{1/3}$

Range: $-\infty < y < \infty$

3 {
- 1 : $2x^3 + e > 0$
- 1 : domain
  Note: 0/1 if 0 is not in the domain
- 1 : range

Note: 0/3 if y is not a logarithmic function of x

6. [1998 AB – NO Calculator] Let f be a function with $f(1)=4$ such that for all points (x, y) on the graph of f the slope is given by $\dfrac{3x^2+1}{2y}$.

(a) Find the slope of the graph of f at the point where $x = 1$.

(b) Write an equation for the line tangent to the graph of f at $x = 1$.

(c) Find $f(x)$ by solving the separable differential equation $\dfrac{dy}{dx} = \dfrac{3x^2+1}{2y}$ with the initial condition $f(1)=4$.

(d) Use your solution from part (c) to find $f(1.2)$.

(a) $\dfrac{dy}{dx} = \dfrac{3x^2+1}{2y}$ $\left.\dfrac{dy}{dx}\right\|_{\substack{x=1\\y=4}} = \dfrac{3+1}{2\cdot 4} = \dfrac{4}{8} = \dfrac{1}{2}$	1: answer
(b) $y - 4 = \frac{1}{2}(x-1)$ $f(1.2) - 4 \approx \frac{1}{2}(1.2-1)$ $f(1.2) \approx 0.1 + 4 = 4.1$	2 { 1: equation of tangent line 1: uses equation to approximate $f(1.2)$
(c) $2y\,dy = (3x^2+1)\,dx$ $\int 2y\,dy = \int (3x^2+1)\,dx$ $y^2 = x^3 + x + C$ $4^2 = 1 + 1 + C$ $14 = C$ $y^2 = x^3 + x + 14$ $y = \sqrt{x^3+x+14}$ is branch with point $(1,4)$ $f(x) = \sqrt{x^3+x+14}$	5 { 1: separates variables 1: antiderivative of dy term 1: antiderivative of dx term 1: uses $y = 4$ when $x = 1$ to pick one function out of a family of functions 1: solves for y 0/1 if solving a linear equation in y 0/1 if no constant of integration Note: max 0/5 if no separation of variables Note: max 1/5 [1-0-0-0-0] if substitutes value(s) for x, y, or dy/dx before antidifferentiation
(d) $f(1.2) = \sqrt{1.2^3 + 1.2 + 14} \approx 4.114$	1: answer, from student's solution to the given differential equation in (c)