Sem 2 Unit 4 – WKST #4 SCORING GUIDE

AP Calculus AB–5

A cubic polynomial function f is defined by

$$f(x) = 4x^3 + ax^2 + bx + k$$

where a, b, and k are constants. The function f has a local minimum at $x = -1$, and the graph of f has a point of inflection at $x = -2$.

(a) Find the values of a and b.

(b) If $\int_0^1 f(x)\,dx = 32$, what is the value of k?

(a) $f'(x) = 12x^2 + 2ax + b$

$f''(x) = 24x + 2a$

$f'(-1) = 12 - 2a + b = 0$

$f''(-2) = -48 + 2a = 0$

$a = 24$

$b = -12 + 2a = 36$

$$5 : \begin{cases} 1 : f'(x) \\ 1 : f''(x) \\ 1 : f'(-1) = 0 \\ 1 : f''(-2) = 0 \\ 1 : a,\ b \end{cases}$$

(b) $\int_0^1 \left(4x^3 + 24x^2 + 36x + k\right) dx$

$= x^4 + 8x^3 + 18x^2 + kx\Big|_{x=0}^{x=1} = 27 + k$

$27 + k = 32$

$k = 5$

$$4 : \begin{cases} 2 : \text{antidifferentiation} \\ \quad <-1> \text{ each error} \\ 1 : \text{expression in } k \\ 1 : k \end{cases}$$

Question 6

The twice-differentiable function f is defined for all real numbers and satisfies the following conditions:

$$f(0) = 2, \quad f'(0) = -4, \text{ and } f''(0) = 3.$$

(a) The function g is given by $g(x) = e^{ax} + f(x)$ for all real numbers, where a is a constant. Find $g'(0)$ and $g''(0)$ in terms of a. Show the work that leads to your answers.

(b) The function h is given by $h(x) = \cos(kx)f(x)$ for all real numbers, where k is a constant. Find $h'(x)$ and write an equation for the line tangent to the graph of h at $x = 0$.

(a) $g'(x) = ae^{ax} + f'(x)$ $g'(0) = a - 4$ $g''(x) = a^2e^{ax} + f''(x)$ $g''(0) = a^2 + 3$	$4: \begin{cases} 1: g'(x) \\ 1: g'(0) \\ 1: g''(x) \\ 1: g''(0) \end{cases}$
(b) $h'(x) = f'(x)\cos(kx) - k\sin(kx)f(x)$ $h'(0) = f'(0)\cos(0) - k\sin(0)f(0) = f'(0) = -4$ $h(0) = \cos(0)f(0) = 2$ The equation of the tangent line is $y = -4x + 2$.	$5: \begin{cases} 2: h'(x) \\ 3: \begin{cases} 1: h'(0) \\ 1: h(0) \\ 1: \text{equation of tangent line} \end{cases} \end{cases}$

Question 6

Let f be the function defined by

$$f(x) = \begin{cases} \sqrt{x+1} & \text{for } 0 \le x \le 3 \\ 5 - x & \text{for } 3 < x \le 5. \end{cases}$$

(a) Is f continuous at $x = 3$? Explain why or why not.

(b) Find the average value of $f(x)$ on the closed interval $0 \le x \le 5$.

(c) Suppose the function g is defined by

$$g(x) = \begin{cases} k\sqrt{x+1} & \text{for } 0 \le x \le 3 \\ mx + 2 & \text{for } 3 < x \le 5, \end{cases}$$

where k and m are constants. If g is differentiable at $x = 3$, what are the values of k and m?

(a) f is continuous at $x = 3$ because

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = 2.$

Therefore, $\lim_{x \to 3} f(x) = 2 = f(3).$

$2: \begin{cases} 1: \text{answers "yes" and equates the values of the left- and right-hand limits} \\ 1: \text{explanation involving limits} \end{cases}$

(b) $$\int_0^5 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^5 f(x)\,dx$$

$$= \frac{2}{3}(x+1)^{3/2}\Big|_0^3 + \left(5x - \frac{1}{2}x^2\right)\Big|_3^5$$

$$= \left(\frac{16}{3} - \frac{2}{3}\right) + \left(\frac{25}{2} - \frac{21}{2}\right) = \frac{20}{3}$$

Average value: $\frac{1}{5}\int_0^5 f(x)\,dx = \frac{4}{3}$

$4: \begin{cases} 1: k\int_0^3 f(x)\,dx + k\int_3^5 f(x)\,dx \\ \quad (\text{where } k \neq 0) \\ 1: \text{antiderivative of } \sqrt{x+1} \\ 1: \text{antiderivative of } 5 - x \\ 1: \text{evaluation and answer} \end{cases}$

(c) Since g is continuous at $x = 3$, $2k = 3m + 2$.

$$g'(x) = \begin{cases} \dfrac{k}{2\sqrt{x+1}} & \text{for } 0 < x < 3 \\ m & \text{for } 3 < x < 5 \end{cases}$$

$\lim_{x \to 3^-} g'(x) = \frac{k}{4}$ and $\lim_{x \to 3^+} g'(x) = m$

Since these two limits exist and g is differentiable at $x = 3$, the two limits are equal. Thus $\frac{k}{4} = m$.

$8m = 3m + 2$; $m = \frac{2}{5}$ and $k = \frac{8}{5}$

$3: \begin{cases} 1: 2k = 3m + 2 \\ 1: \frac{k}{4} = m \\ 1: \text{values for } k \text{ and } m \end{cases}$