Semester 2 Unit 4 – PRACTICE TEST

SCORING GUIDE

1. A particle moves along the y-axis so that its velocity v at time $t \geq 0$ is given by $v(t) = 1 - \tan^{-1}(e^t)$. At time $t = 0$, the particle is at $y = -1$. (Note: $\tan^{-1} x = \arctan x$)

(a) Find the acceleration of the particle at time $t = 2$.

(b) Is the speed of the particle increasing or decreasing at time $t = 2$? Give a reason for your answer.

(c) Find the time $t \geq 0$ at which the particle reaches its highest point. Justify your answer.

(d) Find the position of the particle at time $t = 2$. Is the particle moving toward the origin or away from the origin at time $t = 2$? Justify your answer.

(a) $a(2) = v'(2) = -0.132$ or -0.133	1 : answer
(b) $v(2) = -0.436$ Speed is increasing since $a(2) < 0$ and $v(2) < 0$.	1 : answer with reason
(c) $v(t) = 0$ when $\tan^{-1}(e^t) = 1$ $t = \ln(\tan(1)) = 0.443$ is the only critical value for y. $v(t) > 0$ for $0 < t < \ln(\tan(1))$ $v(t) < 0$ for $t > \ln(\tan(1))$ $y(t)$ has an absolute maximum at $t = 0.443$.	3 : { 1 : sets $v(t) = 0$ 1 : identifies $t = 0.443$ as a candidate 1 : justifies absolute maximum
(d) $y(2) = -1 + \int_0^2 v(t)\,dt = -1.360$ or -1.361 The particle is moving away from the origin since $v(2) < 0$ and $y(2) < 0$.	4 : { 1 : $\int_0^2 v(t)\,dt$ 1 : handles initial condition 1 : value of $y(2)$ 1 : answer with reason

2.

Distance x (cm)	0	1	5	6	8
Temperature $T(x)$ (°C)	100	93	70	62	55

A metal wire of length 8 centimeters (cm) is heated at one end. The table above gives selected values of the temperature $T(x)$, in degrees Celsius (°C), of the wire x cm from the heated end. The function T is decreasing and twice differentiable.

(a) Estimate $T'(7)$. Show the work that leads to your answer. Indicate units of measure.

(b) Write an integral expression in terms of $T(x)$ for the average temperature of the wire. Estimate the average temperature of the wire using a trapezoidal sum with the four subintervals indicated by the data in the table. Indicate units of measure.

(c) Find $\int_0^8 T'(x)\,dx$, and indicate units of measure. Explain the meaning of $\int_0^8 T'(x)\,dx$ in terms of the temperature of the wire.

(d) Are the data in the table consistent with the assertion that $T''(x) > 0$ for every x in the interval $0 < x < 8$? Explain your answer.

(a) $\dfrac{T(8)-T(6)}{8-6} = \dfrac{55-62}{2} = -\dfrac{7}{2}\,°\text{C/cm}$

1 : answer

(b) $\dfrac{1}{8}\int_0^8 T(x)\,dx$

Trapezoidal approximation for $\int_0^8 T(x)\,dx$:

$$A = \frac{100+93}{2}\cdot 1 + \frac{93+70}{2}\cdot 4 + \frac{70+62}{2}\cdot 1 + \frac{62+55}{2}\cdot 2$$

Average temperature $\approx \dfrac{1}{8}A = 75.6875$°C

3 : { 1 : $\frac{1}{8}\int_0^8 T(x)\,dx$; 1 : trapezoidal sum; 1 : answer }

(c) $\int_0^8 T'(x)\,dx = T(8) - T(0) = 55 - 100 = -45$°C

The temperature drops 45°C from the heated end of the wire to the other end of the wire.

2 : { 1 : value; 1 : meaning }

(d) Average rate of change of temperature on $[1, 5]$ is $\dfrac{70-93}{5-1} = -5.75$.

Average rate of change of temperature on $[5, 6]$ is $\dfrac{62-70}{6-5} = -8$.

No. By the MVT, $T'(c_1) = -5.75$ for some c_1 in the interval $(1, 5)$ and $T'(c_2) = -8$ for some c_2 in the interval $(5, 6)$. It follows that T' must decrease somewhere in the interval (c_1, c_2). Therefore T'' is not positive for every x in $[0, 8]$.

2 : { 1 : two slopes of secant lines; 1 : answer with explanation }

Units of °C/cm in (a), and °C in (b) and (c)

1 : units in (a), (b), and (c)

3. Consider the curve given by $x^2 + 4y^2 = 7 + 3xy$.

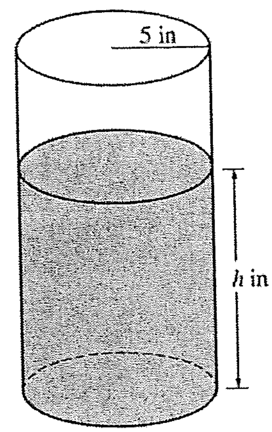
(a) Show that $\frac{dy}{dx} = \frac{3y - 2x}{8y - 3x}$.

(b) Show that there is a point P with x-coordinate 3 at which the line tangent to the curve at P is horizontal. Find the y-coordinate of P.

(c) Find the value of $\frac{d^2y}{dx^2}$ at the point P found in part (b). Does the curve have a local maximum, a local minimum, or neither at the point P? Justify your answer.

(a) $2x + 8yy' = 3y + 3xy'$ $(8y - 3x)y' = 3y - 2x$ $y' = \frac{3y - 2x}{8y - 3x}$	2 : { 1 : implicit differentiation 1 : solves for y'
(b) $\frac{3y - 2x}{8y - 3x} = 0;\ 3y - 2x = 0$ When $x = 3,\ 3y = 6$ $y = 2$ $3^2 + 4 \cdot 2^2 = 25$ and $7 + 3 \cdot 3 \cdot 2 = 25$ Therefore, $P = (3, 2)$ is on the curve and the slope is 0 at this point.	3 : { 1 : $\frac{dy}{dx} = 0$ 1 : shows slope is 0 at $(3, 2)$ 1 : shows $(3, 2)$ lies on curve
(c) $\frac{d^2y}{dx^2} = \frac{(8y - 3x)(3y' - 2) - (3y - 2x)(8y' - 3)}{(8y - 3x)^2}$ At $P = (3, 2)$, $\frac{d^2y}{dx^2} = \frac{(16 - 9)(-2)}{(16 - 9)^2} = -\frac{2}{7}$. Since $y' = 0$ and $y'' < 0$ at P, the curve has a local maximum at P.	4 : { 2 : $\frac{d^2y}{dx^2}$ 1 : value of $\frac{d^2y}{dx^2}$ at $(3, 2)$ 1 : conclusion with justification

4. A coffeepot has the shape of a cylinder with radius 5 inches, as shown in the figure above. Let h be the depth of the coffee in the pot, measured in inches, where h is a function of time t, measured in seconds. The volume V of coffee in the pot is changing at the rate of $-5\pi\sqrt{h}$ cubic inches per second. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)



(a) Show that $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$.

(b) Given that $h = 17$ at time $t = 0$, solve the differential equation $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$ for h as a function of t.

(c) At what time t is the coffeepot empty?

(a) $V = 25\pi h$

$$\frac{dV}{dt} = 25\pi\frac{dh}{dt} = -5\pi\sqrt{h}$$

$$\frac{dh}{dt} = \frac{-5\pi\sqrt{h}}{25\pi} = -\frac{\sqrt{h}}{5}$$

3 :
- 1 : $\frac{dV}{dt} = -5\pi\sqrt{h}$
- 1 : computes $\frac{dV}{dt}$
- 1 : shows result

(b) $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$

$$\frac{1}{\sqrt{h}}\,dh = -\frac{1}{5}\,dt$$

$$2\sqrt{h} = -\frac{1}{5}t + C$$

$$2\sqrt{17} = 0 + C$$

$$h = \left(-\frac{1}{10}t + \sqrt{17}\right)^2$$

5 :
- 1 : separates variables
- 1 : antiderivatives
- 1 : constant of integration
- 1 : uses initial condition $h = 17$ when $t = 0$
- 1 : solves for h

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

(c) $\left(-\frac{1}{10}t + \sqrt{17}\right)^2 = 0$

$$t = 10\sqrt{17}$$

1 : answer

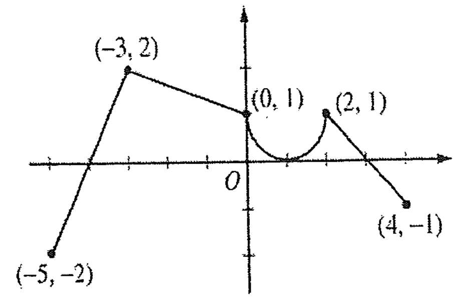
5. The graph of the function f shown above consists of a semicircle and three line segments. Let g be the function given by $g(x) = \int_{-3}^{x} f(t)\,dt$.

(a) Find $g(0)$ and $g'(0)$.

(b) Find all values of x in the open interval $(-5, 4)$ at which g attains a relative maximum. Justify your answer.

(c) Find the absolute minimum value of g on the closed interval $[-5, 4]$. Justify your answer.

(d) Find all values of x in the open interval $(-5, 4)$ at which the graph of g has a point of inflection.



Graph of f

(a) $g(0) = \int_{-3}^{0} f(t)\,dt = \frac{1}{2}(3)(2+1) = \frac{9}{2}$

$g'(0) = f(0) = 1$

$2 : \begin{cases} 1 : g(0) \\ 1 : g'(0) \end{cases}$

(b) g has a relative maximum at $x = 3$.
This is the only x-value where $g' = f$ changes from positive to negative.

$2 : \begin{cases} 1 : x = 3 \\ 1 : \text{justification} \end{cases}$

(c) The only x-value where f changes from negative to positive is $x = -4$. The other candidates for the location of the absolute minimum value are the endpoints.

$g(-5) = 0$

$g(-4) = \int_{-3}^{-4} f(t)\,dt = -1$

$g(4) = \frac{9}{2} + \left(2 - \frac{\pi}{2}\right) = \frac{13 - \pi}{2}$

So the absolute minimum value of g is -1.

$3 : \begin{cases} 1 : \text{identifies } x = -4 \text{ as a candidate} \\ 1 : g(-4) = -1 \\ 1 : \text{justification and answer} \end{cases}$

(d) $x = -3, 1, 2$

2 : correct values
$\langle -1 \rangle$ each missing or extra value

6. Let f be the function defined by $f(x) = k\sqrt{x} - \ln x$ for $x > 0$, where k is a positive constant.

(a) Find $f'(x)$ and $f''(x)$.

(b) For what value of the constant k does f have a critical point at $x = 1$? For this value of k, determine whether f has a relative minimum, relative maximum, or neither at $x = 1$. Justify your answer.

(c) For a certain value of the constant k, the graph of f has a point of inflection on the x-axis. Find this value of k.

(a) $f'(x) = \dfrac{k}{2\sqrt{x}} - \dfrac{1}{x}$

$f''(x) = -\dfrac{1}{4}kx^{-3/2} + x^{-2}$

$2: \begin{cases} 1: f'(x) \\ 1: f''(x) \end{cases}$

(b) $f'(1) = \dfrac{1}{2}k - 1 = 0 \Rightarrow k = 2$

When $k = 2$, $f'(1) = 0$ and $f''(1) = -\dfrac{1}{2} + 1 > 0$.

f has a relative minimum value at $x = 1$ by the Second Derivative Test.

$4: \begin{cases} 1: \text{sets } f'(1) = 0 \text{ or } f'(x) = 0 \\ 1: \text{solves for } k \\ 1: \text{answer} \\ 1: \text{justification} \end{cases}$

(c) At this inflection point, $f''(x) = 0$ and $f(x) = 0$.

$f''(x) = 0 \Rightarrow \dfrac{-k}{4x^{3/2}} + \dfrac{1}{x^2} = 0 \Rightarrow k = \dfrac{4}{\sqrt{x}}$

$f(x) = 0 \Rightarrow k\sqrt{x} - \ln x = 0 \Rightarrow k = \dfrac{\ln x}{\sqrt{x}}$

Therefore, $\dfrac{4}{\sqrt{x}} = \dfrac{\ln x}{\sqrt{x}}$

$\Rightarrow 4 = \ln x$

$\Rightarrow x = e^4$

$\Rightarrow k = \dfrac{4}{e^2}$

$3: \begin{cases} 1: f''(x) = 0 \text{ or } f(x) = 0 \\ 1: \text{equation in one variable} \\ 1: \text{answer} \end{cases}$

Question 3

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	-4	4	2
4	-1	3	6	7

The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table above gives values of the functions and their first derivatives at selected values of x. The function h is given by $h(x) = f(g(x)) - 6$.

(a) Explain why there must be a value r for $1 < r < 3$ such that $h(r) = -5$.

(b) Explain why there must be a value c for $1 < c < 3$ such that $h'(c) = -5$.

(c) Let w be the function given by $w(x) = \int_1^{g(x)} f(t)\,dt$. Find the value of $w'(3)$.

(d) If g^{-1} is the inverse function of g, write an equation for the line tangent to the graph of $y = g^{-1}(x)$ at $x = 2$.

(a) $h(1) = f(g(1)) - 6 = f(2) - 6 = 9 - 6 = 3$
$h(3) = f(g(3)) - 6 = f(4) - 6 = -1 - 6 = -7$
Since $h(3) < -5 < h(1)$ and h is continuous, by the Intermediate Value Theorem, there exists a value r, $1 < r < 3$, such that $h(r) = -5$.

$2: \begin{cases} 1: h(1) \text{ and } h(3) \\ 1: \text{conclusion, using IVT} \end{cases}$

(b) $\dfrac{h(3) - h(1)}{3 - 1} = \dfrac{-7 - 3}{3 - 1} = -5$
Since h is continuous and differentiable, by the Mean Value Theorem, there exists a value c, $1 < c < 3$, such that $h'(c) = -5$..

$2: \begin{cases} 1: \dfrac{h(3) - h(1)}{3 - 1} \\ 1: \text{conclusion, using MVT} \end{cases}$

(c) $w'(3) = f(g(3)) \cdot g'(3) = f(4) \cdot 2 = -2$

$2: \begin{cases} 1: \text{apply chain rule} \\ 1: \text{answer} \end{cases}$

(d) $g(1) = 2$, so $g^{-1}(2) = 1$.
$$\left(g^{-1}\right)'(2) = \frac{1}{g'\left(g^{-1}(2)\right)} = \frac{1}{g'(1)} = \frac{1}{5}$$
An equation of the tangent line is $y - 1 = \frac{1}{5}(x - 2)$.

$3: \begin{cases} 1: g^{-1}(2) \\ 1: \left(g^{-1}\right)'(2) \\ 1: \text{tangent line equation} \end{cases}$

Question 5

t (minutes)	0	2	5	7	11	12
$r'(t)$ (feet per minute)	5.7	4.0	2.0	1.2	0.6	0.5

The volume of a spherical hot air balloon expands as the air inside the balloon is heated. The radius of the balloon, in feet, is modeled by a twice-differentiable function r of time t, where t is measured in minutes. For $0 < t < 12$, the graph of r is concave down. The table above gives selected values of the rate of change, $r'(t)$, of the radius of the balloon over the time interval $0 \le t \le 12$. The radius of the balloon is 30 feet when $t = 5$. (Note: The volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.)

(a) Estimate the radius of the balloon when $t = 5.4$ using the tangent line approximation at $t = 5$. Is your estimate greater than or less than the true value? Give a reason for your answer.

(b) Find the rate of change of the volume of the balloon with respect to time when $t = 5$. Indicate units of measure.

(c) Use a right Riemann sum with the five subintervals indicated by the data in the table to approximate $\int_0^{12} r'(t)\,dt$. Using correct units, explain the meaning of $\int_0^{12} r'(t)\,dt$ in terms of the radius of the balloon.

(d) Is your approximation in part (c) greater than or less than $\int_0^{12} r'(t)\,dt$? Give a reason for your answer.

(a) $r(5.4) \approx r(5) + r'(5)\Delta t = 30 + 2(0.4) = 30.8$ ft
Since the graph of r is concave down on the interval $5 < t < 5.4$, this estimate is greater than $r(5.4)$.

2: { 1 : estimate; 1 : conclusion with reason }

(b) $\frac{dV}{dt} = 3\left(\frac{4}{3}\right)\pi r^2 \frac{dr}{dt}$

$\left.\frac{dV}{dt}\right|_{t=5} = 4\pi(30)^2 2 = 7200\pi \text{ ft}^3/\text{min}$

3: { 2 : $\frac{dV}{dt}$; 1 : answer }

(c) $\int_0^{12} r'(t)\,dt \approx 2(4.0) + 3(2.0) + 2(1.2) + 4(0.6) + 1(0.5)$
$= 19.3$ ft
$\int_0^{12} r'(t)\,dt$ is the change in the radius, in feet, from $t = 0$ to $t = 12$ minutes.

2: { 1 : approximation; 1 : explanation }

(d) Since r is concave down, r' is decreasing on $0 < t < 12$. Therefore, this approximation, 19.3 ft, is less than $\int_0^{12} r'(t)\,dt$.

1 : conclusion with reason

Units of ft^3/min in part (b) and ft in part (c)

1 : units in (b) and (c)

Question 4

A particle moves along the x-axis with position at time t given by $x(t) = e^{-t}\sin t$ for $0 \le t \le 2\pi$.

(a) Find the time t at which the particle is farthest to the left. Justify your answer.

(b) Find the value of the constant A for which $x(t)$ satisfies the equation $Ax''(t) + x'(t) + x(t) = 0$ for $0 < t < 2\pi$.

(a) $x'(t) = -e^{-t}\sin t + e^{-t}\cos t = e^{-t}(\cos t - \sin t)$

$x'(t) = 0$ when $\cos t = \sin t$. Therefore, $x'(t) = 0$ on $0 \le t \le 2\pi$ for $t = \frac{\pi}{4}$ and $t = \frac{5\pi}{4}$.

The candidates for the absolute minimum are at $t = 0, \frac{\pi}{4}, \frac{5\pi}{4}$, and 2π.

t	$x(t)$
0	$e^0 \sin(0) = 0$
$\frac{\pi}{4}$	$e^{-\frac{\pi}{4}}\sin\left(\frac{\pi}{4}\right) > 0$
$\frac{5\pi}{4}$	$e^{-\frac{5\pi}{4}}\sin\left(\frac{5\pi}{4}\right) < 0$
2π	$e^{-2\pi}\sin(2\pi) = 0$

The particle is farthest to the left when $t = \frac{5\pi}{4}$.

$5: \begin{cases} 2: x'(t) \\ 1: \text{sets } x'(t) = 0 \\ 1: \text{answer} \\ 1: \text{justification} \end{cases}$

(b) $x''(t) = -e^{-t}(\cos t - \sin t) + e^{-t}(-\sin t - \cos t)$

$= -2e^{-t}\cos t$

$Ax''(t) + x'(t) + x(t)$

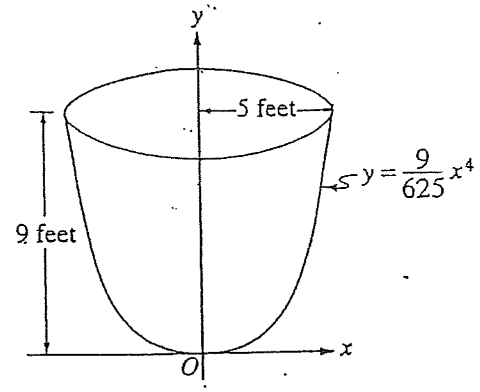
$= A\left(-2e^{-t}\cos t\right) + e^{-t}(\cos t - \sin t) + e^{-t}\sin t$

$= (-2A + 1)e^{-t}\cos t$

$= 0$

Therefore, $A = \frac{1}{2}$.

$4: \begin{cases} 2: x''(t) \\ 1: \text{substitutes } x''(t), x'(t) \text{ and } x(t) \text{ into } Ax''(t) + x'(t) + x(t) \\ 1: \text{answer} \end{cases}$



← OR $\frac{625y}{9} = x^4$

$x = \pm\frac{5}{\sqrt{3}}y^{\frac{1}{4}}$

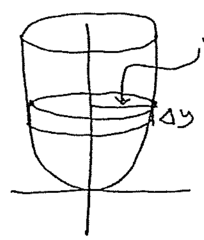
5. An oil storage tank has the shape shown above, obtained by revolving the curve $y = \frac{9}{625}x^4$ from $x = 0$ to $x = 5$ about the y-axis, where x and y are measured in feet. Oil flows into the tank at the constant rate of 8 cubic feet per minute.

(a) Find the volume of the tank. Indicate units of measure.

(b) To the nearest minute, how long would it take to fill the tank if the tank was empty initially?

(c) Let h be the depth, in feet, of oil in the tank. How fast is the depth of the oil in the tank increasing when $h = 4$? Indicate units of measure.

(a)



$r = \frac{5}{\sqrt{3}}y^{\frac{1}{4}}$

$$Vol = \pi\int_0^9 \left(\frac{5}{\sqrt{3}}y^{\frac{1}{4}}\right)^2 dy$$

$$\frac{25\pi}{3}\int_0^9 y^{\frac{1}{2}}\,dy$$

$$\frac{25\pi}{3}\cdot\frac{2}{3}\; y^{\frac{3}{2}}\Big|_0^9$$

$$\frac{50\pi}{9}\left(9^{\frac{3}{2}} - 0^{\frac{3}{2}}\right)$$

$$\frac{50\pi}{9}(27)$$

$$\boxed{150\pi\ ft^3}$$

(b) "Basic" Algebra!

Rate · time = "Quantity"

$8\,\frac{ft^3}{min}\cdot t = 150\pi\,ft^3$

$t = \frac{150\pi\ ft^3}{8\ \frac{ft^3}{min}} = \frac{150\pi}{8}$ min OR $\boxed{\frac{75\pi}{4}\text{ min}}$

(c) Vol at any depth $h = \pi\int_0^h \left(\frac{5}{\sqrt{3}}y^{\frac{1}{4}}\right)^2 dy$

$V = \frac{25\pi}{3}\int_0^h y^{\frac{1}{2}}\,dy$

$V = \frac{50\pi}{9}h^{\frac{3}{2}}$

Find $\frac{dh}{dt}$ when $h=4$ & $\frac{dV}{dt} = 8$

$\frac{dV}{dt} = \frac{50\pi}{9}\cdot\frac{3}{2}h^{\frac{1}{2}}\frac{dh}{dt}$

$\therefore\ 8 = \frac{25\pi}{3}4^{\frac{1}{2}}\frac{dh}{dt}$

$8 = \frac{50\pi}{3}\frac{dh}{dt}$

$\frac{dh}{dt} = \boxed{\frac{12}{25\pi}\ ft/min}$