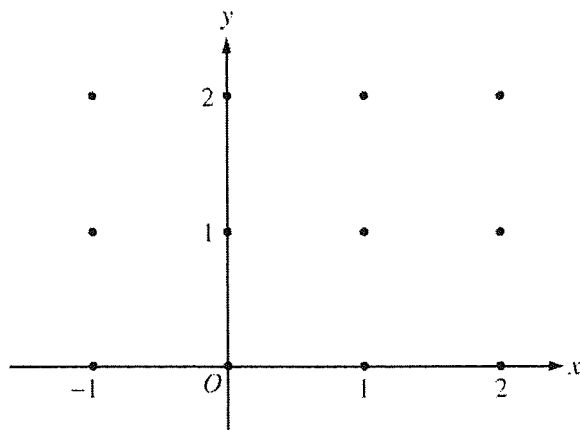


Graphing Calculators are **NOT ALLOWED** for these problems.

Material from the previous two units will be covered on this test.

6. Consider the differential equation $\frac{dy}{dx} = \frac{-xy^2}{2}$. Let $y = f(x)$ be the particular solution to this differential equation with the initial condition $f(-1) = 2$.

- (a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated. (Note: Use the axes provided in the test booklet.)

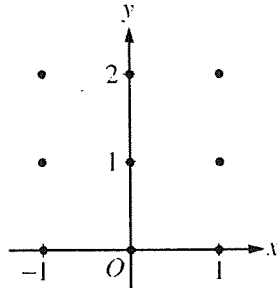


- (b) Write an equation for the line tangent to the graph of f at $x = -1$.
- (c) Find the solution $y = f(x)$ to the given differential equation with the initial condition $f(-1) = 2$.
-

5. Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}x + y - 1$.

(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.

(Note: Use the axes provided in the exam booklet.)



(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y . Describe the region in the xy -plane in which all solution curves to the differential equation are concave up.

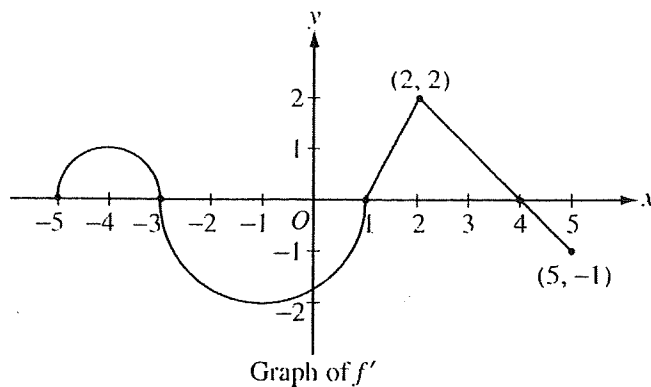
(c) Let $y = f(x)$ be a particular solution to the differential equation with the initial condition $f(0) = 1$. Does f have a relative minimum, a relative maximum, or neither at $x = 0$? Justify your answer.

(d) Find the values of the constants m and b , for which $y = mx + b$ is a solution to the differential equation.

4. Let h be a function defined for all $x \neq 0$ such that $h(4) = -3$ and the derivative of h is given by

$$h'(x) = \frac{x^2 - 2}{x} \text{ for all } x \neq 0.$$

- (a) Find all values of x for which the graph of h has a horizontal tangent, and determine whether h has a local maximum, a local minimum, or neither at each of these values. Justify your answers.
 - (b) On what intervals, if any, is the graph of h concave up? Justify your answer.
 - (c) Write an equation for the line tangent to the graph of h at $x = 4$.
 - (d) Does the line tangent to the graph of h at $x = 4$ lie above or below the graph of h for $x > 4$? Why?
-



4. Let f be a function defined on the closed interval $-5 \leq x \leq 5$ with $f(1) = 3$. The graph of f' , the derivative of f , consists of two semicircles and two line segments, as shown above.
- For $-5 < x < 5$, find all values x at which f has a relative maximum. Justify your answer.
 - For $-5 < x < 5$, find all values x at which the graph of f has a point of inflection. Justify your answer.
 - Find all intervals on which the graph of f is concave up and also has positive slope. Explain your reasoning.
 - Find the absolute minimum value of $f(x)$ over the closed interval $-5 \leq x \leq 5$. Explain your reasoning.
-

Calculator Allowed

2. A particle moves along the x -axis so that its velocity at time t is given by

$$v(t) = -(t + 1) \sin\left(\frac{t^2}{2}\right).$$

At time $t = 0$, the particle is at position $x = 1$.

- Find the acceleration of the particle at time $t = 2$. Is the speed of the particle increasing at $t = 2$? Why or why not?
 - Find all times t in the open interval $0 < t < 3$ when the particle changes direction. Justify your answer.
 - Find the total distance traveled by the particle from time $t = 0$ until time $t = 3$.
 - During the time interval $0 \leq t \leq 3$, what is the greatest distance between the particle and the origin? Show the work that leads to your answer.
-

Calculator Allowed

2. A water tank at Camp Newton holds 1200 gallons of water at time $t = 0$. During the time interval $0 \leq t \leq 18$ hours, water is pumped into the tank at the rate

$$W(t) = 95\sqrt{t} \sin^2\left(\frac{t}{6}\right) \text{ gallons per hour.}$$

During the same time interval, water is removed from the tank at the rate

$$R(t) = 275 \sin^2\left(\frac{t}{3}\right) \text{ gallons per hour.}$$

- (a) Is the amount of water in the tank increasing at time $t = 15$? Why or why not?
- (b) To the nearest whole number, how many gallons of water are in the tank at time $t = 18$?
- (c) At what time t , for $0 \leq t \leq 18$, is the amount of water in the tank at an absolute minimum? Show the work that leads to your conclusion.
- (d) For $t > 18$, no water is pumped into the tank, but water continues to be removed at the rate $R(t)$ until the tank becomes empty. Let k be the time at which the tank becomes empty. Write, but do not solve, an equation involving an integral expression that can be used to find the value of k .
-

Question 6

Consider the differential equation $\frac{dy}{dx} = \frac{-xy^2}{2}$. Let

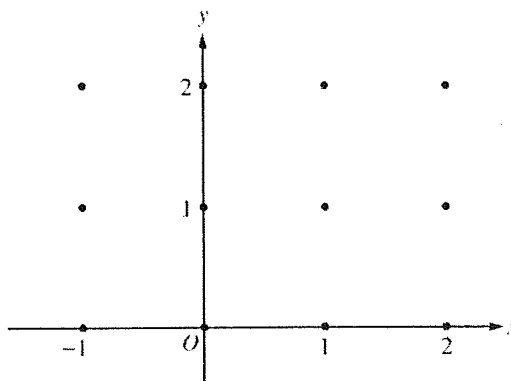
$y = f(x)$ be the particular solution to this differential equation with the initial condition $f(-1) = 2$.

(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated.

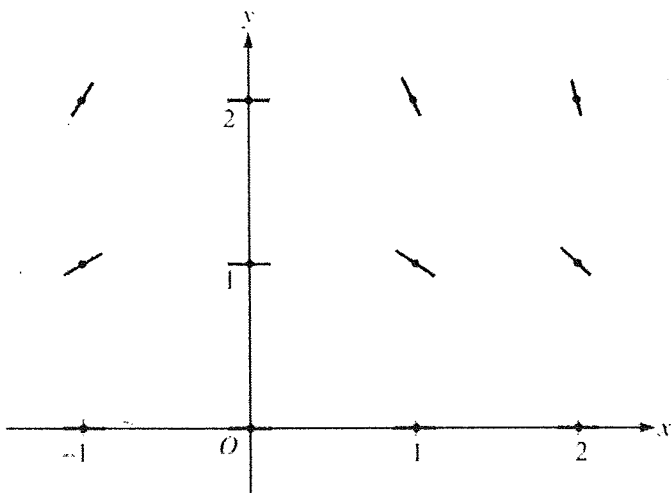
(Note: Use the axes provided in the test booklet.)

(b) Write an equation for the line tangent to the graph of f at $x = -1$.

(c) Find the solution $y = f(x)$ to the given differential equation with the initial condition $f(-1) = 2$.



(a)



(b) Slope = $\frac{-(-1)4}{2} = 2$
 $y - 2 = 2(x + 1)$

(c) $\frac{1}{y^2} dy = -\frac{x}{2} dx$
 $-\frac{1}{y} = -\frac{x^2}{4} + C$
 $-\frac{1}{2} = -\frac{1}{4} + C; C = -\frac{1}{4}$
 $y = \frac{1}{\frac{x^2}{4} + \frac{1}{4}} = \frac{4}{x^2 + 1}$

2 : $\begin{cases} 1 : \text{zero slopes} \\ 1 : \text{nonzero slopes} \end{cases}$

1 : equation

6 : $\begin{cases} 1 : \text{separates variables} \\ 2 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max 3/6 [1-2-0-0-0] if no constant of integration

Note: 0/6 if no separation of variables

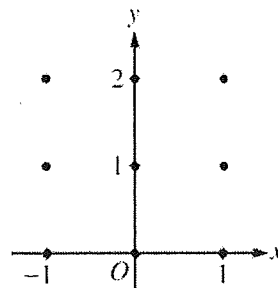
Question 5

Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}x + y - 1$.

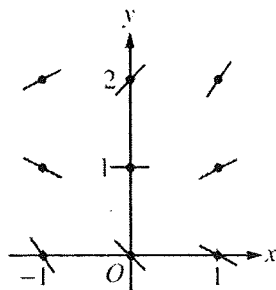
- (a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.

(Note: Use the axes provided in the exam booklet.)

- (b) Find $\frac{d^2y}{dx^2}$ in terms of x and y . Describe the region in the xy -plane in which all solution curves to the differential equation are concave up.
- (c) Let $y = f(x)$ be a particular solution to the differential equation with the initial condition $f(0) = 1$. Does f have a relative minimum, a relative maximum, or neither at $x = 0$? Justify your answer.
- (d) Find the values of the constants m and b , for which $y = mx + b$ is a solution to the differential equation.



(a)



(b) $\frac{d^2y}{dx^2} = \frac{1}{2} + \frac{dy}{dx} = \frac{1}{2}x + y - \frac{1}{2}$

Solution curves will be concave up on the half-plane above the line $y = -\frac{1}{2}x + \frac{1}{2}$.

(c) $\left. \frac{dy}{dx} \right|_{(0,1)} = 0 + 1 - 1 = 0$ and $\left. \frac{d^2y}{dx^2} \right|_{(0,1)} = 0 + 1 - \frac{1}{2} > 0$

Thus, f has a relative minimum at $(0, 1)$.

- (d) Substituting $y = mx + b$ into the differential equation:

$$m = \frac{1}{2}x + (mx + b) - 1 = \left(m + \frac{1}{2}\right)x + (b - 1)$$

Then $0 = m + \frac{1}{2}$ and $m = b - 1$: $m = -\frac{1}{2}$ and $b = \frac{1}{2}$.

2 : Sign of slope at each point and relative steepness of slope lines in rows and columns.

3 : $\begin{cases} 2 : \frac{d^2y}{dx^2} \\ 1 : \text{description} \end{cases}$

2 : $\begin{cases} 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

2 : $\begin{cases} 1 : \text{value for } m \\ 1 : \text{value for } b \end{cases}$

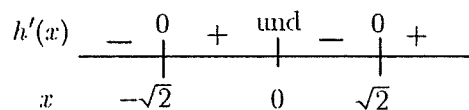
Question 4

Let h be a function defined for all $x \neq 0$ such that $h(4) = -3$ and the derivative of h is given

by $h'(x) = \frac{x^2 - 2}{x}$ for all $x \neq 0$.

- (a) Find all values of x for which the graph of h has a horizontal tangent, and determine whether h has a local maximum, a local minimum, or neither at each of these values. Justify your answers.
- (b) On what intervals, if any, is the graph of h concave up? Justify your answer.
- (c) Write an equation for the line tangent to the graph of h at $x = 4$.
- (d) Does the line tangent to the graph of h at $x = 4$ lie above or below the graph of h for $x > 4$? Why?

(a) $h'(x) = 0$ at $x = \pm\sqrt{2}$



Local minima at $x = -\sqrt{2}$ and at $x = \sqrt{2}$

(b) $h''(x) = 1 + \frac{2}{x^2} > 0$ for all $x \neq 0$. Therefore, the graph of h is concave up for all $x \neq 0$.

(c) $h'(4) = \frac{16 - 2}{4} = \frac{7}{2}$

$$y + 3 = \frac{7}{2}(x - 4)$$

(d) The tangent line is below the graph because the graph of h is concave up for $x > 4$.

$$4 : \begin{cases} 1 : x = \pm\sqrt{2} \\ 1 : \text{analysis} \\ 2 : \text{conclusions} \\ \quad < -1 > \text{not dealing with} \\ \quad \text{discontinuity at } 0 \end{cases}$$

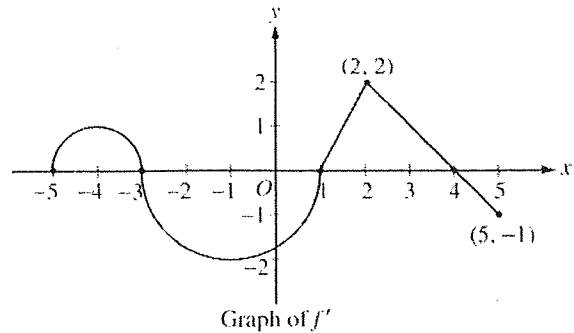
$$3 : \begin{cases} 1 : h''(x) \\ 1 : h''(x) > 0 \\ 1 : \text{answer} \end{cases}$$

1 : tangent line equation

1 : answer with reason

Question 4

Let f be a function defined on the closed interval $-5 \leq x \leq 5$ with $f(1) = 3$. The graph of f' , the derivative of f , consists of two semicircles and two line segments, as shown above.



- For $-5 < x < 5$, find all values x at which f has a relative maximum. Justify your answer.
- For $-5 < x < 5$, find all values x at which the graph of f has a point of inflection. Justify your answer.
- Find all intervals on which the graph of f is concave up and also has positive slope. Explain your reasoning.
- Find the absolute minimum value of $f(x)$ over the closed interval $-5 \leq x \leq 5$. Explain your reasoning.

- (a) $f'(x) = 0$ at $x = -3, 1, 4$
 f' changes from positive to negative at -3 and 4 .
 Thus, f has a relative maximum at $x = -3$ and at $x = 4$.

2 : { 1 : x-values
 1 : justification

- (b) f' changes from increasing to decreasing, or vice versa, at $x = -4, -1, \text{ and } 2$. Thus, the graph of f has points of inflection when $x = -4, -1, \text{ and } 2$.

2 : { 1 : x-values
 1 : justification

- (c) The graph of f is concave up with positive slope where f' is increasing and positive: $-5 < x < -4$ and $1 < x < 2$.

2 : { 1 : intervals
 1 : explanation

- (d) Candidates for the absolute minimum are where f' changes from negative to positive (at $x = 1$) and at the endpoints ($x = -5, 5$).

3 : { 1 : identifies $x = 1$ as a candidate
 1 : considers endpoints
 1 : value and explanation

$$f(-5) = 3 + \int_1^{-5} f'(x) dx = 3 - \frac{\pi}{2} + 2\pi > 3$$

$$f(1) = 3$$

$$f(5) = 3 + \int_1^5 f'(x) dx = 3 + \frac{3 \cdot 2}{2} - \frac{1}{2} > 3$$

The absolute minimum value of f on $[-5, 5]$ is $f(1) = 3$.

Question 2

A particle moves along the x -axis so that its velocity at time t is given by

$$v(t) = -(t+1)\sin\left(\frac{t^2}{2}\right).$$

At time $t = 0$, the particle is at position $x = 1$.

- (a) Find the acceleration of the particle at time $t = 2$. Is the speed of the particle increasing at $t = 2$? Why or why not?
- (b) Find all times t in the open interval $0 < t < 3$ when the particle changes direction. Justify your answer.
- (c) Find the total distance traveled by the particle from time $t = 0$ until time $t = 3$.
- (d) During the time interval $0 \leq t \leq 3$, what is the greatest distance between the particle and the origin? Show the work that leads to your answer.

(a) $a(2) = v'(2) = 1.587$ or 1.588

$$v(2) = -3\sin(2) < 0$$

Speed is decreasing since $a(2) > 0$ and $v(2) < 0$.

$$2 : \begin{cases} 1 : a(2) \\ 1 : \text{speed decreasing} \\ \text{with reason} \end{cases}$$

(b) $v(t) = 0$ when $\frac{t^2}{2} = \pi$

$$t = \sqrt{2\pi} \text{ or } 2.506 \text{ or } 2.507$$

Since $v(t) < 0$ for $0 < t < \sqrt{2\pi}$ and $v(t) > 0$ for $\sqrt{2\pi} < t < 3$, the particle changes directions at $t = \sqrt{2\pi}$.

$$2 : \begin{cases} 1 : t = \sqrt{2\pi} \text{ only} \\ 1 : \text{justification} \end{cases}$$

(c) Distance = $\int_0^3 |v(t)| dt = 4.333$ or 4.334

$$3 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$$

(d) $\int_0^{\sqrt{2\pi}} v(t) dt = -3.265$

$$x(\sqrt{2\pi}) = x(0) + \int_0^{\sqrt{2\pi}} v(t) dt = -2.265$$

Since the total distance from $t = 0$ to $t = 3$ is 4.334 , the particle is still to the left of the origin at $t = 3$. Hence the greatest distance from the origin is 2.265 .

$$2 : \begin{cases} 1 : \pm \text{ (distance particle travels} \\ \text{while velocity is negative)} \\ 1 : \text{answer} \end{cases}$$

Question 2

A water tank at Camp Newton holds 1200 gallons of water at time $t = 0$. During the time interval $0 \leq t \leq 18$ hours, water is pumped into the tank at the rate

$$W(t) = 95\sqrt{t} \sin^2\left(\frac{t}{6}\right) \text{ gallons per hour.}$$

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 (b) To the nearest whole number, how many gallons of water are in the tank at time $t = 18$?
 (c) At what time t , for $0 \leq t \leq 18$, is the amount of water in the tank at an absolute minimum? Show the work that leads to your conclusion.
 (d) For $t > 18$, no water is pumped into the tank, but water continues to be removed at the rate $R(t)$ until the tank becomes empty. Let k be the time at which the tank becomes empty. Write, but do not solve, an equation involving an integral expression that can be used to find the value of k .

- (a) No; the amount of water is not increasing at $t = 15$ since $W(15) - R(15) = -121.09 < 0$.

1 : answer with reason

- (b) $1200 + \int_0^{18} (W(t) - R(t)) dt = 1309.788$
 1310 gallons

3 : $\begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

- (c) $W(t) - R(t) = 0$
 $t = 0, 6.4948, 12.9748$

t (hours)	gallons of water
0	1200
6.495	525
12.975	1697
18	1310

3 : $\begin{cases} 1 : \text{interior critical points} \\ 1 : \text{amount of water is least at} \\ \quad t = 6.494 \text{ or } 6.495 \\ 1 : \text{analysis for absolute minimum} \end{cases}$

The values at the endpoints and the critical points show that the absolute minimum occurs when $t = 6.494$ or 6.495 .

- (d) $\int_{18}^k R(t) dt = 1310$

2 : $\begin{cases} 1 : \text{limits} \\ 1 : \text{equation} \end{cases}$