Material from the previous units will be covered on this test.

A graphing calculator is required for some problems or parts of problems.

1. A particle moves along the y-axis so that its velocity $v$ at time $t \geq 0$ is given by $v(t) = 1 - \tan^{-1}(e^t)$. At time $t = 0$, the particle is at $y = -1$. (Note: $\tan^{-1} x = \arctan x$)

   $v'(0) = -1 \quad v'(0) = -\frac{e^t}{1 + (e^t)^2}$

   (a) Find the acceleration of the particle at time $t = 2$.

   (b) Is the speed of the particle increasing or decreasing at time $t = 2$? Give a reason for your answer.

   (c) Find the time $t \geq 0$ at which the particle reaches its highest point. Justify your answer.

   (d) Find the position of the particle at time $t = 2$. Is the particle moving toward the origin or away from the origin at time $t = 2$? Justify your answer.

\[
\begin{align*}
\text{v}'(2) &= -1.132 < 0 \\
v(2) &= -0.426 < 0
\end{align*}
\]

\[
\text{The speed at } t=2 \text{ is increasing since } v'(2) < 0 \text{ and } v(2) < 0.
\]

\[
\text{Find where } v(t) = 0 \text{ and } t = 0
\]

\[
y(0) = 1
\]

\[
y(0.493) = y(0) + \int_0^{0.493} v(t) \, dt = 5.7532
\]

\[
at t = 0.493 \text{ since } y(0.493) > y(0)
\]

\[
y(2) = y(0) + \int_0^2 v(t) \, dt = -1.760
\]

\[
\text{Particle moves away from } (0,0) \text{ since } \frac{x}{-t} \quad v(2) < 0 \\
y(2) < 0
\]

\[
\text{Find distance on } 0 \leq t \leq 2. \text{ Dist } = \int_0^2 |v(t)| \, dt
\]

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2. A metal wire of length 8 centimeters (cm) is heated at one end. The table above gives selected values of the temperature \( T(x) \), in degrees Celsius (\(^\circ\)C), of the wire \( x \) cm from the heated end. The function \( T \) is decreasing and twice differentiable.

(a) Estimate \( T'(7) \). Show the work that leads to your answer. Indicate units of measure.

(b) Write an integral expression in terms of \( T(x) \) for the average temperature of the wire. Estimate the average temperature of the wire using a trapezoidal sum with the four subintervals indicated by the data in the table. Indicate units of measure.

(c) Find \( \int_0^8 T'(x) \, dx \), and indicate units of measure. Explain the meaning of \( \int_0^8 T'(x) \, dx \) in terms of the temperature of the wire.

(d) Are the data in the table consistent with the assertion that \( T''(x) > 0 \) for every \( x \) in the interval \( 0 < x < 8 \)? Explain your answer.

\[
\text{Average Temperature} = \frac{1}{8-0} \int_0^8 T(x) \, dx
\]

\[
= \frac{1}{8} \left[ \frac{1}{3} \left( 100 + 93 + 70 + 62 \right) + \frac{1}{3} \left( 70 + 62 \right) \right] + \frac{1}{3} \left( 70 + 62 \right)
\]

\[
= \frac{1}{8} \left[ 93(5) + 62(3) \right]
\]

WRITE ALL WORK IN THE EXAM BOOKLET.

\[
\int_0^8 T'(x) \, dx = T(x) \Big|_0^8 = T(8) - T(0)
\]

\[
= 55 - 100
\]

\[
= -45 \, ^\circ\text{C}
\]

 Français

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3. Consider the curve given by \( x^2 + 4y^2 = 7 + 3xy \).

(a) Show that \( \frac{dy}{dx} = \frac{3y - 2x}{8y - 3x} \).

(b) Show that there is a point \( P \) with \( x \)-coordinate 3 at which the line tangent to the curve at \( P \) is horizontal. Find the \( y \)-coordinate of \( P \).

(c) Find the value of \( \frac{d^2y}{dx^2} \) at the point \( P \) found in part (b). Does the curve have a local maximum, a local minimum, or neither at the point \( P \)? Justify your answer.

\[
\frac{dy}{dx} \Big|_{(3,2)} = \frac{-2(16 - 9) - (6 - 2)(-3)}{(16 - 9)^2}
\]

\( 2x + 8y \frac{dy}{dx} = 3y^2 + 2 \times 1 \cdot \frac{dy}{dx} \)

\( \frac{dy}{dx} \frac{dy}{dx} - 3 \times \frac{dy}{dx} = 3y - 2x \)

\( \frac{d}{dx} (3y - 2x) = 3y - 2x \)

\( \frac{d}{dx} \frac{dy}{dx} = \frac{3y - 2x}{8y - 3x} \)

It's good!

WRITE ALL WORK IN THE EXAM BOOKLET.

\( \frac{d^2y}{dx^2} \bigg|_{(3,2)} = \frac{-2}{7} \)

Since \( \frac{d^2y}{dx^2} < 0 \) and \( \frac{dy}{dx} \Big|_{(3,2)} = 0 \), by 2nd Deriv Test, there is a local max at \( (3,2) \).
4. A coffeepot has the shape of a cylinder with radius 5 inches, as shown in the figure above. Let \( h \) be the depth of the coffee in the pot, measured in inches, where \( h \) is a function of \( t \), measured in seconds. The volume \( V \) of coffee in the pot is changing at the rate of \( -5\pi \sqrt{h} \) cubic inches per second. (The volume \( V \) of a cylinder with radius \( r \) and height \( h \) is \( V = \pi r^2 h \).)

(a) Show that \( \frac{dh}{dt} = -\frac{\sqrt{h}}{5} \).

(b) Given that \( h = 17 \) at time \( t = 0 \), solve the differential equation \( \frac{dh}{dt} = -\frac{\sqrt{h}}{5} \) for \( h \) as a function \( t \).

(c) At what time \( t \) is the coffeepot empty?

\[ h^2 = \frac{1}{5} t + 2\sqrt{17} \]

\[ h^2 = \frac{7}{10} t + 2\sqrt{17} \]

\[ h = \left( \frac{\sqrt{7}}{10} t + 2\sqrt{17} \right)^2 \]

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5. The graph of the function $f$ shown above consists of a semicircle and three line segments. Let $g$ be the function given by

$$g(x) = \int_{-3}^{x} f(t) \, dt.$$ 

(a) Find $g(0)$ and $g'(0)$.

$$g(0) = \int_{-3}^{0} f(t) \, dt = \frac{1}{2}(2+1) = \frac{3}{2}$$

(b) Find the values of $x$ in the open interval $(-5, 4)$ at which $g$ attains a relative maximum. Justify your answer.

At $x = 3$, since $g'$ changes from positive to negative, $g$ attains a relative maximum.

(c) Find the absolute minimum value of $g$ on the closed interval $[-5, 4]$. Justify your answer.

$$g(4) = g(-3) - \int_{-3}^{4} f(t) \, dt = 0 - 1 = -1$$

WRITE ALL WORK IN THE EXAM BOOKLET.

d) $x = -3, x = 1, x = 2$ since $g'(x)$ changes from either inc to dec or dec to inc at $x = 3, 1, 2$.

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6. Let $f$ be a function defined by $f(x) = k\sqrt{x} - \ln x$ for $x > 0$, where $k$ is a positive constant.

(a) Find $f'(x)$ and $f''(x)$.

$$f'(x) = \frac{1}{2} k x^{-\frac{1}{2}} - \frac{1}{x}$$

$$f''(x) = -\frac{1}{4} k x^{-\frac{3}{2}} - x^{-2}$$

(b) For what value of the constant $k$ does $f$ have a critical point at $x = 1$? For this value of $k$, determine whether $f$ has a relative minimum, relative maximum, or neither at $x = 1$. Justify your answer.

(c) For a certain value of the constant $k$, the graph of $f$ has a point of inflection on the $x$-axis. Find this value of $k$.

\[ \begin{align*}
\frac{1}{2} k & = 1 \\
k & = 2
\end{align*} \]

WRITE ALL WORK IN THE EXAM BOOKLET.

Since $f'(1) = 0$ and $f''(1) > 0$, by the 2nd Deriv Test, $f$ has a local min at $x = 1$.

END OF EXAM
3. The functions $f$ and $g$ are differentiable for all real numbers, and $g$ is strictly increasing. The table above gives values of the functions and their first derivatives at selected values of $x$. The function $h$ is given by $h(x) = f(g(x)) - 6$.

(a) Explain why there must be a value $r$ for $1 < r < 3$ such that $h(r) = -5$.

(b) Explain why there must be a value $c$ for $1 < c < 3$ such that $h'(c) = -5$.

(c) Let $w$ be the function given by $w(x) = \int_{1}^{x} f(t) dt$. Find the value of $w'(3)$.

(d) If $g^{-1}$ is the inverse function of $g$, write an equation for the line tangent to the graph of $y = g^{-1}(x)$ at $x = 2$.

\[ h(3) - h(1) = \frac{3 - 1}{3 - 1} = \frac{-7 - 3}{2} = -5 \]

\[ h'(3) = \frac{f'(3) g'(3)}{f(g(3))} = \frac{4 \cdot 2}{6} = \frac{8}{6} = \frac{4}{3} \]

Since $h$ is differentiable on $1 < c < 3$ and $g$ is continuous on $1 \leq c \leq 3$, $h'(c) = -5$, then by MVT, there exists a value $c$ on $1 < c < 3$ such that $h'(c) = -5$.

\[ m = \frac{f'(3) g'(3)}{f(g(3))} = \frac{4 \cdot 2}{6} = \frac{8}{6} = \frac{4}{3} \]

\[ y - 1 = \frac{4}{3} (x - 2) \]

\[ m = \frac{dy}{dx} \bigg|_{x=2} = \frac{4}{3} \]
5. The volume of a spherical hot air balloon expands as the air inside the balloon is heated. The radius of the balloon, in feet, is modeled by a twice-differentiable function \( r \) of time \( t \), where \( t \) is measured in minutes. For \( 0 < t < 12 \), the graph of \( r \) is concave down. The table above gives selected values of the rate of change, \( r'(t) \), of the radius of the balloon over the time interval \( 0 \leq t \leq 12 \). The radius of the balloon is 30 feet when \( t = 5 \).

(Note: The volume of a sphere of radius \( r \) is given by \( V = \frac{4}{3} \pi r^3 \))

(a) Estimate the radius of the balloon when \( t = 5.4 \) using the tangent line approximation at \( t = 5 \). Is your estimate greater than or less than the true value? Give a reason for your answer.

(b) Find the rate of change of the volume of the balloon with respect to time when \( t = 5 \). Indicate units of measure.

(c) Use a right Riemann sum with the five subintervals indicated by the data in the table to approximate \( \int_0^{12} r'(t) \, dt \). Using correct units, explain the meaning of \( \int_0^{12} r'(t) \, dt \) in terms of the radius of the balloon.

(d) Is your approximation in part (c) greater than or less than \( \int_0^{12} r'(t) \, dt \)? Give a reason for your answer.

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\[ r(5.4) = r(5) + r'(5) \cdot \Delta t \]

\[ = 30 + 2 \cdot (0.4) \]

\[ = 30.8 \text{ ft} \]

Since \( r \) is concave down.

\[ \int_0^{12} r'(t) \, dt \]

**c.**

Find \( \frac{dV}{dt} \) when \( t = 5 \).

Given \( \frac{dr}{dt} = 2.0 \)

from table.

\[ \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \]

\[ \frac{dV}{dt} = 4\pi (30)^2 \cdot 2 \]

\[ = 7200\pi \text{ ft}^3/\text{min} \]

\[ = 11482 \text{ ft}^3/\text{min} \]